Indian Statistical Institute, Bangalore
B. Math (Hons.) First Year
First Semester - Linear Algebra I

Semestral Exam — Date: 06th December 2022
Maximum marks: 50 — Duration: 3 hours

Answer any five, each question carries 10 marks

1. (i) Prove that a linearly independent subset is contained in a basis (**Marks: 5**).

   (ii) Find a complementary subspace of $W = \{v \in \mathbb{R}^n \mid \sum i v_i = 0\}$.

2. (i) $r_0(A) = r_0(A^2)$ if and only if $C(A) \cap N(A) = \{0\}$.

   (ii) Prove that $N(AA^*) = N(A^*)$ and $r_0(AA^*) = r_0(A)$ (**Marks: 5**).

3. (i) Let A and B be of order $n \times n$. Prove that $r_0(AB) \geq r_0(A) + r_0(B) - n$.

   (ii) Prove that $A_1 A_2 \cdots A_N = 0$ implies $\sum \dim(N(A_i)) \geq n$ where A_i are $n \times n$-matrices (**Marks: 5**)..

4. (i) For $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \end{pmatrix}$, find all B such that $AB = I$, using g-inverse of A.

   (ii) Prove that all g-inverses of A are of the form $A^g + U - A^g AUAA^g$ where A^g is a given g-inverse of A (**Marks: 5**).

5. (i) Prove that $A(A^*A)^g A^*$ is hermitian and invariant for any g-inverse of AA^*.

   (ii) Find all g-inverses of $A = \begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ of rank 3 (**Marks: 5**).

6. (i) Prove that a reflexive g-inverse A^g of A such that AA^g and $A^g A$ are hermitian is unique (**Marks 5**).

   (ii) Prove that pivot columns are independent and their span is the range.

7. (i) For any matrix A, prove that there is a permutation matrix P such that $PA = LU$ where L and U are lower and upper triangular matrices.

   (ii) Can P in (i) be identity matrix always? Justify your answer (**Marks: 2**).

   (iii) For $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 3 \\ -6 & -1 & 2 \end{pmatrix}$, can $A = LU$ where L and U are lower and upper triangular matrices (**Marks: 3**).